WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 3rd Semester Examination, 2020, held in 2021

MTMACOR05T-MATHEMATICS (CC5)

THEORY OF REAL FUNCTIONS

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Does $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ exist?

(b) Evaluate: $\lim\limits_{x\to 3}\left([x]-\left[\dfrac{x}{3}\right]\right)$

(c) Show that $\lim\limits_{x\to 0} \sin\dfrac{1}{x}$ does not exist.

(d) Examine the continuity of

$$f(x)=\begin{cases} x & ; \quad 1\le x<2 \\ 3x+4 & ; \quad x\ge 2 \end{cases}$$

at $x=2$.

(e) Determine $f(0)$ so that the function

$$f(x)=\frac{x^2-x}{x} \quad ; \quad x\neq 0$$

is continuous at $x=0$.

(f) Show that $f:\mathbb{R}\longrightarrow\mathbb{R}$ defined by

$$f(x)=\begin{cases} 2x & ; \quad x\in\mathbb{Q} \\ 1-x & ; \quad x\notin\mathbb{Q} \end{cases}$$

is continuous only at $\dfrac{1}{3}$ and discontinuous at all other points.

(g) Examine whether the function defined by

$$f(x)=\begin{cases} x\cos\dfrac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

is differentiable at $x=0$.

(h) Examine validity of Rolle's theorem for the function

$$f(x)=x(x+3)e^{-x/2}, \quad x\in[-3,\ 0].$$

Also, verify the conclusion of Rolle's theorem for this function, if possible.

(i) Verify Lagrange's mean value theorem for the following function:

$$f(x)=1+x^{2/3}\ ,\ \ \forall\, x\in[-8,\ 1].$$

(j) Show that $f(x)=x^3-6x^2+24x+4$ has neither a maximum nor a minimum.

2. (a) Let $f:D\to\mathbb{R}$ where $D\subseteq\mathbb{R}$ and let $\lim_{x\to a} f(x)=l\neq 0$. Show that there is a neighbourhood N of a so that f has the same sign as l in $(N-\{a\})\cap D$. 5

(b) Show that $\lim_{x\to\infty}\dfrac{x-[x]}{x}=0$. 3

3. (a) Let $f:[a,\ b]\to\mathbb{R}$ be a continuous function. If $f(a)$ and $f(b)$ have opposite signs, then show that there is at least one $c\in(a,b)$ such that $f(c)=0$. 5

(b) Show that there exists a root of $x+x\log x-3=0$ in (1, 3). 3

4. (a) Let $f:[0,1]\to\mathbb{R}$ be a continuous function such that $f(x)\in\mathbb{Q},\ \forall\, x\in[0,1]$. Show that f is a constant function on [0, 1]. 3

(b) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Let 5

$$\sup_{x\in[a,\,b]} f(x)=M \quad\text{and}\quad \inf_{x\in[a,\,b]} f(x)=m$$

Show that there is at least one $c\in[a,b]$ such that $f(c)=M$ and there is at least one $d\in[a,b]$ such that $f(d)=m$.

5. (a) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Show that f is uniformly continuous. 5

(b) Show that the following function is uniformly continuous: 3

$$f(x)=\sqrt{x}\ ,\ \forall\, x\in[1,\infty).$$

6. (a) Let $f:I\to\mathbb{R}$, where I is an interval in $\mathbb{R}$. Let $c\in I$. Show that f is differentiable at c if and only if there is a function $\varphi:I\to\mathbb{R}$ continuous at c satisfying. 5

$$f(x)-f(c)=\varphi(x)(x-c)\ ,\ \ \forall\, x\in I.$$

Further show that in this case $\varphi(c)=f'(c)$.

(b) Show that $f(x)$ is differentiable at $x=0$ but the derived function f' is not continuous at $x=0$ where 3

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ,\quad x\neq 0\\ 0 & ,\quad x=0\end{cases}$$

7. (a) Let $f:I\to\mathbb{R}$ and $g:J\to\mathbb{R}$ be such that Image $f\subseteq J$, where $I,\ J$ are intervals in $\mathbb{R}$. Let f be differentiable at $c\in I$ and g be differentiable at $f(c)=d\in J$. Show that $g\circ f:I\to\mathbb{R}$ is differentiable at c and 5

$$(g\circ f)'(c)=g'(f(c))\,f'(c)$$

(b) With proper justification prove that 3

$$\frac{d}{dx}(\sin^{-1}(x))=\frac{1}{\sqrt{1-x^2}}\ ,\ \ \forall x,\ -1<x<1.$$

8. (a) State and prove Rolle's theorem. 1+4

(b) Show that between any two distinct real roots of $e^x \sin x + 1 = 0$ there is at least one real root of $\tan x + 1 = 0$. 3

9. (a) Is Mean value theorem applicable to the function $f(x) = |x|$ on [–1, 1]? 2

(b) If a real valued function f on an interval I be derivable and bounded on I, then prove that f is uniformly continuous on I. 3

(c) Use Mean value theorem to prove that 3

$$\frac{1}{x} < \frac{1}{\log(1+x)} < 1 + \frac{1}{x}.$$

10.(a) Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable in (a, b). Prove that if $f'(t) > 0$, $\forall\, t \in (a, b)$, then f is strictly increasing on $[a, b]$. 4

(b) Prove that 4

$$f(x) = \left(1 - \frac{1}{x}\right)^x, \ \forall\, x > 1$$

is increasing on (1, ∞).

11.(a) State and prove Cauchy's Mean Value theorem. 2+4

(b) Let f be a continuous function defined on [0, 1] which is differentiable on (0, 1). Show that 2

$$f(1) - f(0) = \frac{f'(x)}{2x}$$

has at least one solution in (0, 1).

12.(a) Write with proper justification, Maclaurin's infinite series expansion for 5

$$f(x) = \log(1+x), \quad -1 < x \le 1$$

(b) Let f, f' be continuous on $[a, b]$ and f'' exist in (a, b). Show that there exists at least one point $c \in (a, b)$ such that 3

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(c).$$

13.(a) A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis and the minor axis at P and Q respectively. Show that the least value of PQ is $a + b$. 5

(b) Show that the semi-vertical angle of a right circular cone of minimum possible surface and of given volume is $\sin^{-1}\left(\frac{1}{3}\right)$. 3

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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